Remark on Hamilton Cycles: "Given graph G, find a hamilton cycle if it exists or prove there is no such cycle." is an "NP-Complete problem"

↳ "Yes" answers are easy to verify

↳ Answers are "hard" to produce...

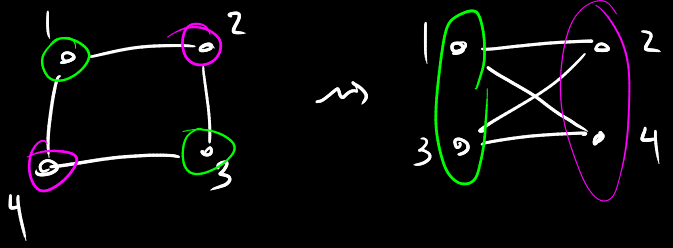
Exercise: Show Petersen graph has no Hamilton cycle...

↳ Post a proof on the website. :).

Bipartite Graphs

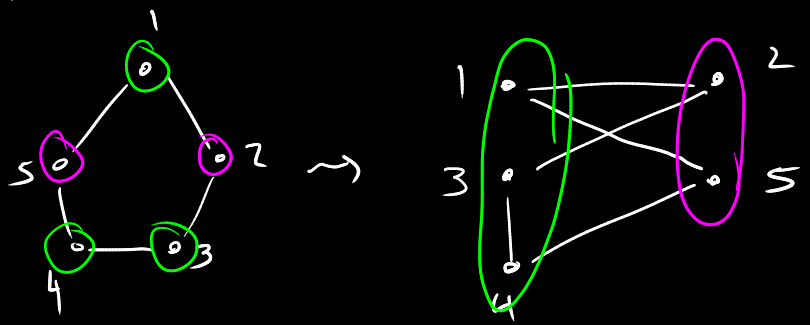
Defn: A graph G is bipartite when there is a bipartition $V(G) = L \sqcup R$ of the vertices of G such that for all $e \in E(G)$, the ends of e belong to different parts of the bipartition.

Ex:



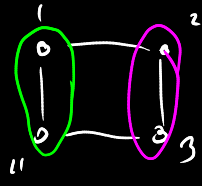
$\rightsquigarrow$ C_4 is bipartite.

Ex:



the given L and R do not form a bipartition of C_5...

NB:

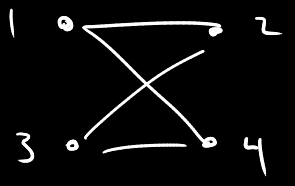


$\rightsquigarrow$ Not a bipartition of C_4, but... C_4 was bipartite...

Claim: C_5 is NOT bipartite... why? ← on hold for a minute.

Defn: Let L and R be disjoint sets. the Complete bipartite graph on (L, R) is $K_{L,R} = (L \sqcup R, \{lr : l \in L, r \in R\})$.

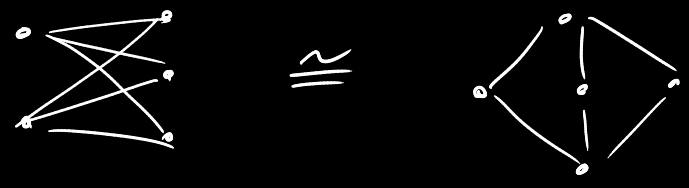
Ex: If $L = \{1,3\}$, $R = \{2,4\}$, then $K_{L,R}$:

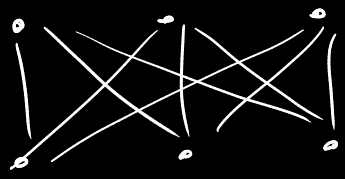
$K_{L,R} =$ 

$= C_4$

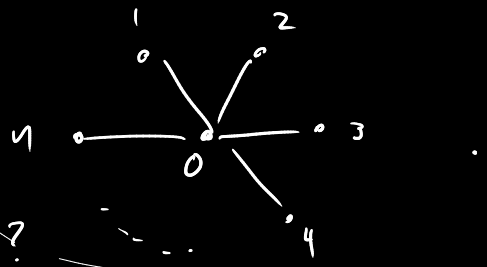
Convention: Often I write "$K_{m,n}$" meaning $K_{L,R}$ for disjoint sets L, R with $\#L = m$, $\#R = n$.

With this convention, we see C_4 and $K_{2,2}$ describe the same gph.

Ex: $K_{2,3}$: 

Ex: $K_{3,3}$ 

Ex: $K_{0,n} = \overline{K}_n$ is the "edgeless graph" on n vertices.

Ex: $K_{1,n}$ is a "star" 

Q: How do we tell if G is bipartite?

Prop: Let G be a graph. The following are equivalent:

1. G is bipartite.
2. G is a subgraph of a complete bipartite graph.
3. G has no odd cycle subgraphs.
4. Every closed walk in G has even length.

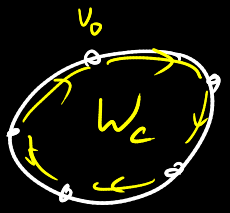
Sketch: Let G be a graph.

(1) $\Rightarrow$ (2): If G is bipartite, then $V(G) = L \sqcup R$ for some L, R with $e \in E(G)$ implies $\#(e \cap L) = 1 = \#(e \cap R)$.

Hence $G \subseteq K_{L,R}$ (because $e \in E(G)$ implies $e = \{l, r\}$ for some $l \in L$ and $r \in R$, so $E(G) \subseteq E(K_{L,R})$).

② ⇒ ④: We show that every closed walk in $K_{L,R}$ has even length. Let $W = (v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n)$ be a closed walk in $K_{L,R}$. Each e_i "crosses" from L to R, so each e_i is flanked by an $l \in L$ and $r \in R$. As the walk W is closed, $v_0 = v_n$, so there are an even number of edges in the walk (lest $v_0 \in L$ and $v_n \in R$ contradicting $L \cap R = \emptyset$).

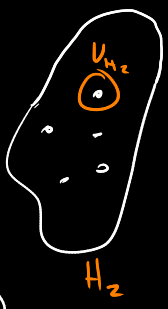
④ ⇒ ③: Every cycle subgraph in G determines a walk by starting from some vertex and walking along new edges until arriving back at the start vertex. But this walk necessarily has even length and traverses every edge exactly once. Hence C has an even number of edges.

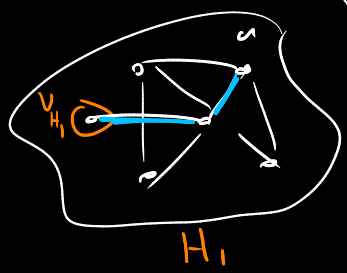


③ ⇒ ①: Assuming every cycle of G has even length, split the vertices of G into two parts via the following procedure:

Ⓐ In each component H of G choose one vertex v_H.

Ⓑ Let L be the vertices u of G with a shortest path from u to some v_H having even length, and let R be the rest.





Claim: $L \sqcup R$ is a vertex bipartition of G.

If not,

Picture: